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On Weak Chebyshev Subspaces. I. Equioscillation of the Error in Approximation

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This paper is a generalization of the result obtained by F. Deutsch, G. Nürnberger, and I. Singer (1980, *Pacific J. Math.* **88**, 9–31). It is shown that if Q is a locally compact totally ordered space, and N is an n-dimensional subspace of $C_0(Q)$, then N is a weak Chebyshev subspace if and only if for each $f \in C_0(Q)$, there is $g \in N$ such that $\|f-g\| = d(f, N)$ and $(f-g)$ equioscillates at $(n+1)$ points.

1. INTRODUCTION

The closed subset C of the Banach space X is said to be "proximinal" in X if for each $x \in X$ there is $y_0 \in C$ such that $d(x, C) = \|x - y_0\|$, where $d(x, C)$ is the distance of x from C, that is,

$$d(x, C) = \inf\{\|x - z\|;\ z \in C\}.$$

If C is proximinal in X, then the set valued function $P_C: X \Rightarrow 2^C$ defined by $P_C(x) = \{y \in C;\ d(x, C) = \|x - y\|\}$ is called the "metric projection" from X onto C, and the continuous function $g: X \to C$ is called a "continuous selection" for the metric projection P_C if $g(x) \in P_C(x)$ for each $x \in X$.

If Q is a compact Hausdorff space, then $C(Q)$ is the Banach space of all continuous real valued functions defined on Q, and if Q is a locally compact Hausdorff space then $C_0(Q)$ is the Banach space of all continuous real valued functions defined on Q and "vanishing at infinity"; that is, if $f \in C_0(Q)$, then for all $\varepsilon > 0$ the set $\{q \in Q;\ |f(q)| \geqslant \varepsilon\}$ is compact. The

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norm defined on $C(Q)$ and $C_0(Q)$ is the supremum norm; that is, $\|f\| = \sup\{|f(q)|; q \in Q\}$.

The n-dimensional subspace N of $C(Q)$ is called a Chebyshev subspace if each $g \neq 0$ in N has at most $(n-1)$ zeros in Q. By the Haar theorem, the n-dimensional subspace N of $C(Q)$ is a Chebyshev subspace if and only if for each $f \in C(Q)$, the set $P_N(f)$ is a singleton. Using this property one can easily show that, if N is an n-dimensional Chebyshev subspace of $C(Q)$, then the metric projection P_N is a continuous function from $C(Q)$ onto N. This property fails if N is not a Chebyshev subspace, and in this case the metric projection P_N may even fail to have a continuous selection. However, there are strong restrictions on the existence of Chebyshev subspaces; for example, Schoenberg and Yang [7] proved that if $n \geqslant 2$, and $C(Q)$ contains an n-dimensional Chebyshev subspace, then Q is homeomorphic to a subset of the circle.

Many other authors have tried to find some weaker conditions which insure the existence of a continuous selection for the metric projection. In this area of approximation theory the "totally ordered spaces" play an important role. Brown [1] showed that if $n \geqslant 2$ and $C(Q)$ contains an n-dimensional subspace N, for which the metric projection P_N has a continuous selection, and no $g \neq 0$ in N vanishes at a nonempty open interval in Q, then either Q is homeomorphic to the circle or Q is a totally ordered space. Another important concept is the concept of "weak Chebyshev subspaces." If Q is a locally compact totally ordered space, then the n-dimensional subspace N of $C_0(Q)$ is called a weak Chebyshev subspace of $C_0(Q)$ if each g in N has at most $(n-1)$ changes of sign; that is, there do not exist $(n+1)$ points $x_1 < x_2 < \cdots < x_{n+1}$ in Q such that $g(x_i)\, g(x_{i+1}) < 0$ for each $i = 1, 2, \ldots, n$. Nürnberger [6] showed that if N is an n-dimensional subspace of $C[a, b]$ such that the metric projection P_N has a continuous selection, then N is a weak Chebyshev subspace. Other important properties that may be satisfied by some n-dimensional subspace of $C_0(Q)$ and that are essential in the proof of the results concerning weak Chebyshev subspaces are mentioned in Definition 1.1.

1.1. Definition. Let Q be a locally compact totally ordered space and let N be an n-dimensional subspace of $C_0(Q)$. The subspace N may or may not possess one of the following properties:

wc_1: N is weak Chebyshev.

wc_2: For each basis $\{g_1, g_2, \ldots, g_n\}$ of N and points $t_1 < t_2 < \cdots < t_n$ in Q and $s_1 < s_2 < \cdots < s_n$ in Q it is always true that

$$\det[g_i(t_j)]\, \det[g_i(s_j)] \geqslant 0.$$

wc_3: For each $x_1 < x_2 < \cdots < x_{n-1}$ in Q, there is $g \neq 0$ in N such that

$$\begin{aligned} g(x_i) &= 0 && \text{for } i = 1, 2, ..., n-1, \\ (-1)^i g(x) &\geqslant 0 && \text{for } x \in \{x \in Q; x_i \leqslant x \leqslant x_{i+1}\} \text{ and } i = 1, 2, ..., n-2, \\ g(x) &\geqslant 0 && \text{for } x \in \{x \in Q; x \leqslant x_1\}, \\ (-1)^{n-1} g(x) &\geqslant 0 && \text{for } x \in \{x \in Q; x \geqslant x_{n-1}\}. \end{aligned}$$

wc_4: For each $f \in C_0(Q)$ there is $g \in N$ such that $\|f - g\| = d(f, N)$ and $\|f - g\|$ equioscillates at $(n+1)$ points of Q; that is, there are $x_1 < x_2 < \cdots < x_{n+1}$ in Q and $\varepsilon = \pm 1$ such that

$$(-1)^i (f-g)(x_i) = \varepsilon \|f - g\|, \qquad \text{for} \quad i = 1, 2, ..., n+1.$$

Jones and Karlovitz [4] proved that if Q is a real compact interval then the four properties are equivalent on any n-dimensional subspace of $C(Q)$, and Deutsch, Nürnberger, and Singer [3] obtained the same result in the case when Q is any locally compact subset of the real numbers. However, in both cases the proof depends on the properties of the real numbers and cannot be generalized.

The properties wc_1, wc_2, and wc_3 are algebraic and the following proposition clarifies the relationship among them. The proof of this proposition is the same as the proof of Lemma 4.1 of Zielke [9, p. 12].

1.2. PROPOSITION. *Let Q be a locally totally ordered space, and let N be an n-dimensional subspace of $C_0(Q)$. Then the three properties wc_1, wc_2, and wc_3 for N are equivalent.*

In this paper it is shown that for any locally compact totally ordered space Q, the property wc_4 is equivalent to each of the other three properties on any n-dimensional subspace of $C_0(Q)$. The proof is simple and depends on the topological properties of Q. At the end of Section 2, it is shown that there is a compact totally ordered space Q, which is not homeomorphic to any subset of the real numbers, such that for each positive integer $n \geqslant 1$, $C(Q)$ contains an n-dimensional weak Chebyshev subspace.

The rest of this section will cover some definitions and notations that will be used frequently in this paper. In this paper, "Q is a totally ordered space" means that Q is a totally ordered set, with the order topology defined on it. The intervals $[x, y]$, (x, y) and the termonilogies $-\infty$ and $+\infty$ have their ordinary meanings. For example, $(-\infty, x_0) = \{x \in Q; x < x_0\}$. If Q is a locally compact totally ordered space, then the function $f \in C_0(Q)$ is said to "equioscillate" at n points of Q, if there are $q_1 < q_2, < \cdots < q_n$ in Q and $\varepsilon = \pm 1$ such that

$$(-1)^i f(q_i) = \varepsilon \|f\| \qquad \text{for} \quad i = 1, 2, ..., n.$$

1.3. DEFINITION. Let Q be a locally compact totally ordered space, and let N be an n-dimensional subspace of $C_0(Q)$. Then the subspace N may or may not possess one of the following properties:

c_1: Each $g \neq 0$ in N has at most $(n-1)$ zeros in Q.

c_2: If $\{g_1, g_2, ..., g_n\}$ is a basis for N, $t_1 < t_2 < \cdots < t_n$ in Q and $s_1 < s_2 < \cdots < s_n$ in Q then

$$\det[g_i(t_j)] \cdot \det[g_i(s_j)] > 0.$$

c_3: If $x_1 < x_2 < \cdots < x_{n-1}$ in Q, then there is g in N, such that

$$g(x_i) = 0 \qquad \text{for} \quad i = 1, 2, ..., n-1,$$

$$(-1)^i g(x) > 0 \qquad \text{for} \quad x \in (x_i, x_{i+1}) \text{ and } i = 0, 1, ..., n-1,$$

where $x_0 = -\infty$ and $x_n = +\infty$.

It is well known that if Q is a compact real interval then c_1, c_2, and c_3 are equivalent for any n-dimensional subspace of $C(Q)$.

1.4. LEMMA. *Let Q be a locally compact totally ordered space, and let N be an n-dimensional subspace of $C_0(Q)$. If N has the property c_2 then N has the property c_1 and the property c_3.*

Proof. $c_2 \Rightarrow c_1$: If N does not have the property c_1 then there is $g \neq 0$ in N and $t_1 < t_2 < \cdots < t_n$ in Q, such that $g(t_i) = 0$ for each $i = 1, 2, ..., n$. But then for any basis $\{g_1, g_2, ..., g_n\}$ of N, $\det[g_i(t_j)] = 0$ so N does not have the property c_2.

$c_2 \Rightarrow c_3$: Let $\{g_1, g_2, ..., g_n\}$ be a basis for N, if $x_1 < x_2 < \cdots < x_{n-1}$ are any $(n-1)$ points of Q, then define $g \in N$ as

$$g(x) = \begin{vmatrix} g_1(x_1) & \cdots & g_1(x_{n-1}) & g_1(x) \\ g_2(x_1) & \cdots & g_2(x_{n-1}) & g_2(x) \\ \vdots & & \vdots & \vdots \\ g_n(x_1) & \cdots & g_n(x_{n-1}) & g_n(x) \end{vmatrix}.$$

Since N has the property c_2 it follows that g has the required properties.

1.5. PROPOSITION. *Let Q be a compact totally ordered space, and let N be an n-dimensional subspace of $C(Q)$ that has the property c_2. Then for* *each $f \in C(Q)$ and each $g \in N$, $\|f-g\| = d(f, N)$ if and only if $(f-g)$ equioscillates at $(n+1)$ points of Q.*

Proof. By Lemma 1.4, the subspace N is a Chebyshev subspace of $C(Q)$; thus the result follows from Singer [8, Theorem 1.4, p. 182] and the fact that N has the property c_2.

2. The Fourth Property of Weak Chebyshev Subspaces

In this section it is shown first that when Q is finite, then each weak Chebyshev subspace of $C(Q)$ has the property wc_4. This is established in Lemma 2.3. In Lemma 2.4 the proof is generalized to any locally compact totally ordered space. In Lemma 2.8 it is shown that every finite dimensional subspace of $C_0(Q)$ that has the property wc_4 is a weak Chebyshev subspace. At the end of this section an example showing that the generalization in this paper is not vacuous is given.

2.1. Proposition. *Let $Q = \{x_1, x_2, ..., x_m\}$ be a finite totally ordered set such that $x_1 < x_2 < \cdots < x_m$, let $n \leqslant m$ be a positive integer, and let N be an n-dimensional weak Chebyshev subspace of $C(Q)$ generated by $\{g_1, g_2, ..., g_n\}$. For each $\sigma > 0$ and each $j = 1, 2, ..., m$, define*

$$g_k^\sigma(x_j) = \sum_{i=1}^{m} e^{-\sigma(i-j)^2} \cdot g_k(x_i), \qquad \text{for} \quad k = 1, 2, ..., n.$$

Then

(1) *For each $k = 1, 2, ..., n$, g_k^σ converges uniformly to g_k as $\sigma \to \infty$.*

(2) *For each $\sigma > 0$, the subspace N^σ of $C(Q)$ generated by $\{g_1^\sigma, g_2^\sigma, ..., g_n^\sigma\}$ has the property c_2.*

Proof. It follows from Proposition 1.2 of Karlin [5, p. 220], and the fact that N has the property wc_2.

2.2. Lemma. *Let Q be a locally compact totally ordered space, and let N be an n-dimensional weak Chebyshev subspace of $C_0(Q)$. For each $f \in C_0(Q)$ and $g \in N$, if $(f-g)$ equioscillates at $(n+1)$ points of Q, then $\|f-g\| = d(f, N)$.*

Proof. Assume not; then there is $h \in N$, such that $\|f-h\| < \|f-g\|$. Since $f-g$ equioscillates at $(n+1)$ points, it follows that there are $x_1 < x_2 < \cdots < x_{n+1}$ in Q and $\varepsilon = \pm 1$, such that for each $k = 1, 2, ..., n+1$

$$(-1)^k (f-g)(x_k) = \varepsilon \|f-g\|.$$

Let $r=h-g$. Then for each $k=1, 2, ..., n+1$,

$$\begin{aligned}\varepsilon(-1)^k r(x_k) &= \varepsilon(-1)^k (f-g)(x_k) - \varepsilon(-1)^k (f-h)(x_k)\\ &= \|f-g\| - \varepsilon(-1)^k (f-h)(x_k)\\ &\geqslant \|f-g\| - \|f-h\| > 0.\end{aligned}$$

Thus r has more than $(n-1)$ changes of sign. Therefore N is not a weak Chebyshev subspace, which is a contradiction.

2.3. LEMMA. *Let $Q=\{x_1, x_2, ..., x_m\}$ be a finite totally ordered space such that $x_1 < x_2 < \cdots < x_m$, let $n \leqslant m-1$ be a positive integer, and let N be an n-dimensional weak Chebyshev subspace of $C(Q)$. Then for each $f \in C(Q)$, there is $g \in N$ such that $\|f-g\| = d(f, N)$ and $f-g$ equioscillates at $(n+1)$ points of Q.*

Proof. If $f \in N$ then $d(f, N) = \|f-f\| = 0$ and 0 equioscillates. So one may assume that $f \notin N$.

Let $\{g_1, g_2, ..., g_n\}$ be a basis for N. For each $i \geqslant 1$, let $\sigma_i = i$. By Proposition 2.1 the n-dimensional subspace N^i of $C(Q)$ generated by $\{g_1^i, g_n^i, ..., g_n^i\}$ has the property c_2. Thus by Proposition 1.5, if g^i is the unique best approximation to f from N^i, then $f-g^i$ equioscillates at $(n+1)$ points; that is, there is $\varepsilon_i = \pm 1$ and $x_1^i < x_2^i < \cdots < x_{n+1}^i$ in Q, such that for each $k=1, 2, ..., n+1$,

$$(-1)^k (f-g^i)(x_k^i) = \varepsilon_i \|f-g^i\|.$$

Since $g^i \in N^i$, it follows that there are $\lambda_1^i, \lambda_2^i, ..., \lambda_n^i$ in R such that $g^i = \sum_{k=1}^n \lambda_k^i g_k^i$. Using the fact that $g_k^i \to g_k$ uniformly on Q, $\|g^i\| \leqslant 2 \|f\|$. Using the fact that $g_1, g_2, ..., g_n$ are linearly independent, it follows that the sequence $\{(\lambda_1^i, ..., \lambda_n^i)\}_{i=1}^\infty$ is bounded in l_n^∞. Therefore, the sequence $\{(x_1^i, ..., x_{n+1}^i, \varepsilon_i, \lambda_1^i, ..., \lambda_n^i)\}_{i=1}^\infty$ has a convergent subsequence in $(\prod_{k=1}^{n+1} Q) \times l_{n+1}^\infty$.

Without loss of generality assume that $\{(x_1^i, ..., x_{n+1}^i, \varepsilon_i, \lambda_1^i, ..., \lambda_n^i)\}_{i=1}^\infty$ converges to $(x_1, ..., x_{n+1}, \varepsilon, \lambda_1, ..., \lambda_n)$, and that $\varepsilon_i = \varepsilon$ for each $i \geqslant 1$. It is clear that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n+1}$. Let $g = \sum_{k=1}^n \lambda_k g_k$ in N; by Lemma 2.2 it is enough to show that $f-g$ equioscillates at $(n+1)$ points.

By Proposition 2.1 for each $k=1, 2, ..., n$, $g_k^i \to g_k$. Thus since $(\lambda_1^i, ..., \lambda_n^i) \to (\lambda_1, ..., \lambda_n)$, it follows that

$$\lim_{i\to\infty} \|f-g^i\| = \|f-g\|.$$

Furthermore $(x_1^i, ..., x_{n+1}^i) \to (x_1, ..., x_{n+1})$, so for each $k=1, 2, ..., n+1$,

$$\lim_{i\to\infty} (f-g^i)(x_k^i) = (f-g)(x_k).$$

Thus for each $k = 1, 2, ..., n+1$, $\varepsilon \|f-g\| = \lim_{i\to\infty} \varepsilon_i \|f-g^i\| = (-1)^k \lim_{i\to\infty} (f-g^i)(x_k^i) = (-1)^k (f-g)(x_k)$. Since $\|f-g\| \neq 0$, it follows that $x_1 < x_2 < \cdots < x_{n+1}$ and $(f-g)$ equioscillates at $x_1 < x_2 < \cdots < x_{n+1}$.

2.4. Lemma. *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional weak Chebyshev subspace of $C_0(Q)$. Then for each $f \in C_0(Q)$, there is $g \in N$ such that $\|f-g\| = d(f, N)$ and $f-g$ equioscillates at $(n+1)$ points of Q.*

Proof. If $f \in N$, then $d(f, N) = 0$ and $f-f=0$ equioscillates, so one may assume that $f \notin N$.

Let $\{g_1, g_2, ..., g_n\}$ be a basis for N. Since Q contains at least $(n+1)$ points, it follows that one can find a compact subset Q' of Q satisfying the following properties:

(a) $g_1, ..., g_n$ and f are linearly independent over Q'.

(b) For each $x \in Q \backslash Q'$, and each $g \in N$ with $\|g\| \leqslant 2\|f\|$, the inequality $|(f-g)(x)| < d(f, N)$ holds.

For each positive integer $i \geqslant 1$, let $\varepsilon_i = 1/i$. By the compactness of Q', and the continuity of the functions $g_1, g_2, ..., g_n$ and f, one can find a family $\{A_i\}_{i=1}^{\infty}$ of finite subsets of Q' satisfying the following properties:

(1) $A_i \subseteq A_{i+1}$, for each $i \geqslant 1$.

(2) For each $i \geqslant 1$, the functions $g_1, ..., g_n$ and f are linearly independent over A_i.

(3) For each $i \geqslant 1$ and each $x \in Q'$, there is $y \in A_i$, such that for each $h \in \{g_1, ..., g_n, f\}$ the inequality $|h(x) - h(y)| < \varepsilon_i$ holds.

For each $h \in C_0(Q)$, let $h^i = h\,|_{A_i}$. Then it is obvious that the n-dimensional subspace N^i generated by $\{g_1^i, ..., g_n^i\}$ is a weak Chebyshev subspace in $C(A_i)$. By Lemma 2.3 there is $g^i = \sum_{k=1}^{n} \lambda_k^i g_k^i$ in N^i such that $\|f^i - g^i\| = d(f^i, N^i)$ and $f^i - g^i$ equioscillates at $n+1$ points; that is, there is $\delta_i = \pm 1$ and $x_1^i < x_2^i < \cdots < x_{n+1}^i$ in A_i, such that for each $k = 1, 2, ..., n+1$,

$$(-1)(f^i - g^i)(x_k^i) = \delta_i \|f^i - g^i\|.$$

Using the definition of A_i and g_k^i, the fact that $\|g^i\| \leqslant 2\|f\|$, and the fact that $g_1, ..., g_n$ are linearly independent, it follows that the sequence $\{(\lambda_1^i, ..., \lambda_n^i)\}_{i=1}^{\infty}$ is bounded in l_n^{∞}. Thus the sequence $\{(x_1^i, ..., x_{n+1}^i, \delta_i, \lambda_1^i, ..., \lambda_n^i)\}_{i=1}^{\infty}$ has a cluster point $(x_1, ..., x_{n+1}, \delta, \lambda_1, ..., \lambda_n)$ in $(\prod_{k=1}^{n+1} Q) \times l_{n+1}^{\infty}$.

Let $g=\sum_{k=1}^{n} \lambda_k g_k$ in N. Then as in Lemma 2.3 one can show that $f-g$ equioscillates at $x_1, ..., x_{n+1}$. Therefore, since N is a weak Chebyshev subspace of $C_0(Q)$, it follows by Lemma 2.2 that $\|f-g\| = d(f, N)$.

In the following definition, $B(Q)$ is the set of all bounded real valued functions defined on Q.

2.5. DEFINITION. Let Q be a locally compact totally ordered space, and let $x_1 < x_2 < \cdots < x_n$ be n points in Q. Then the function $f \in B(Q)$ is said to alternate between the n points $x_1, ..., x_n$ if there is $\varepsilon = \pm 1$ such that

$$\varepsilon(-1)^i f(x) \geqslant 0, \qquad \text{for} \quad x \in (x_i, x_{i+1}) \text{ and } i = 0, 1, ..., n.$$

where $x_0 = -\infty$ and $x_{n+1} = \infty$.

In Definition 2.5 the function f can take any value at the points $x_1, ..., x_n$, and some of the sets (x_i, x_{i+1}) for $i = 0, 1, ..., n$ might be empty. It is clear that if f alternates between n points then f has at most n changes of sign, but the converse is not generally true.

2.6. LEMMA. *Let Q be a locally compact totally ordered space, let $n \geqslant 1$ be a positive integer, and let $x_1 < x_2 < \cdots < x_{n-1}$ be $n-1$ elements in Q. Let $f \in B(Q)$ be a bounded function that alternates between the points $x_1, ..., x_{n-1}$. If $g \in C_0(Q)$ is such that $f-g$ equioscillates at $n+1$ points, then there is $x \in Q \setminus \{x_1, ..., x_{n-1}\}$ such that $|g(x)| \geqslant \|f-g\|$.*

Proof. The proof is by induction. If $n=1$ then there are $y_1 < y_2$ in Q and $\varepsilon = \pm 1$ such that $(-1)^k (f-g)(y_k) = \varepsilon \|f-g\|$ for $k = 1, 2$. Thus

$$(-1)^{k+1} g(y_k) = \varepsilon \|f-g\| + (-1)^{k+1} f(y_k), \qquad \text{for} \quad k = 1, 2.$$

Since f does not change sign on Q, it follows that $f(y_1) f(y_2) \geqslant 0$, so there is $x \in \{y_1, y_2\}$ such that $|g(x)| = \|f-g\| + |f(x)|$; that is, $|g(x)| \geqslant \|f-g\|$. Let $n \geqslant 1$ be a positive integer and assume that the hypothesis is true for each positive integer $k \leqslant n$. It will be shown that the hypothesis is true for $k = n+1$. Assume that f alternates between the points $x_1, ..., x_n$, where $x_1 < x_2 < \cdots < x_n$ in Q, and $f-g$ equioscillates at $y_1, ..., y_{n+2}$, where $y_1 < y_2 < \cdots < y_{n+2}$ in Q. Then there is $\varepsilon = \pm 1$ such that

$$(-1)^k (f-g)(y_k) = \varepsilon \|f-g\| \qquad \text{for} \quad k = 1, 2, ..., n+2.$$

(a) Assume that $y_{k+1} = x_k$ for $k = 1, 2, ..., n$. Then $y_1 < x_1$ and $g(y_1) = \varepsilon \|f-g\| + f(y_1)$. If $|g(y_1)| \geqslant \|f-g\|$, then there is nothing to prove. If $|g(y_1)| < \|f-g\|$, then $-\varepsilon f(y_1) > 0$. Since f does not change sign on the set $\{x \in Q; x < x_1\}$, it follows that $-\varepsilon f(x) \geqslant 0$ for each $x < x_1$. Thus

$\varepsilon(-1)^{n+1} f(x) \geqslant 0$ for each $x > x_n$. Since $y_{n+2} > x_n$, it follows that $\varepsilon(-1)^{n+1} f(y_{n+2}) \geqslant 0$. But

$$(-1)^{n+1} g(y_{n+2}) = \varepsilon \|f-g\| + (-1)^{n+1} f(y_{n+2}),$$

so $|g(y_{n+2})| = \|f-g\| + |f(y_{n+2})| \geqslant \|f-g\|$.

(b) Assume that there is $k_0 \in \{1, 2, ..., n\}$ such that $y_{k_0+1} \neq x_{k_0}$. Then $k_0 \leqslant n$. If $y_{k_0+1} < x_{k_0}$, let $Q' = \{x \in Q; x \leqslant y_{k_0+1}\}$ and $\{x_1, ..., x_h\} = Q' \cap \{x_1, ..., x_{n-1}\}$. The function $f|_{Q'}$ alternates between the points $x_1, ..., x_h$, $(f-g)|_{Q'}$ equioscillates at $y_1, ..., y_{k_0+1}$ and $h \leqslant k_0 - 1$. Since the hypothesis is true for $k = h+1 \leqslant k_0$, it follows that there is $x \in Q' \setminus \{x_1, ..., x_{n-1}\}$ such that $|g(x)| \geqslant \|(f-g)|_{Q'}\| = \|f-g\|$. If $y_{k_0+1} > x_{k_0}$, let $Q' = \{x \in Q; x \geqslant y_{k_0+1}\}$ and let $\{x_1, ..., x_h\} = Q' \cap \{x_1, ..., x_{n-1}\}$. Since $h+1 \leqslant n-k_0$, it follows that one can show that there is $x \in Q' \setminus \{x_1, ..., x_{n-1}\}$ such that $|g(x)| \geqslant \|(f-g)|_{Q'}\| = \|f-g\|$.

2.7. Lemma. *Let Q be a locally compact totally ordered space, and let N be an n-dimensional subspace of $C_0(Q)$. If N has the property wc_4, then for each $x_1 < x_2 < \cdots < x_{n-1}$ in Q and each $\{\sigma_1, ..., \sigma_{n-1}\}$ with $\sigma_i = \pm 1$, there is $g \in N$ satisfying the following properties*:

(1) *$g(x_k) = 0$ if x_k is not an isolated point in Q, and $\sigma_k g(x_k) \geqslant 0$ if x_k is an isolated point in Q.*

(2) *$(-1)^k g(x) \geqslant 0$, for $x \in (x_k, x_{k+1})$ and $k = 0, 1, 2, ..., n-1$, where $x_0 = -\infty$ and $x_n = \infty$.*

(3) *There is $x \in Q \setminus \{x_1, ..., x_{n-1}\}$ such that $g(x) \neq 0$.*

Proof. Let $f: Q \to R$ be a bounded function defined on Q as follows:

(a) $(-1)^k f(x) = 1$, for $x \in (x_k, x_{k+1})$ and $k = 0, 1, 2, ..., n-1$.

(b) $|f(x_k)| = 1$ for $k = 1, 2, ..., n-1$, and the sign of $f(x_k)$ satisfies the property that sign $f(x_k) = \sigma_k$ if x_k is an isolated point in Q, and f is not continuous at x_k if x_k is not isolated in Q.

If Q is compact and the points $x_1, ..., x_{n-1}$ are isolated, then $f \in C_0(Q)$. Thus there is $g \in N$ such that $\|f-g\| = d(f. N) \leqslant \|f\| = 1$, and $f-g$ equioscillates at $n+1$ points. It is obvious that for each $x \in Q$, $g(x) \cdot f(x) \geqslant 0$, so applying Lemma 2.6, one can show that g has the required properties.

If Q is not compact or if $\{x_1, ..., x_{n-1}\}$ are not all isolated, then let Σ be the set of all the compact subsets of Q of the form $A = A_0 \cup A_1 \cup \cdots \cup A_{n-1}$, where A_i is a compact subset of Q satisfying the following properties:

(i) $A_i \subseteq [x_i, x_{i+1}]$ for $1 = 0, 1, 2, ..., n-1$.

(ii) $A_i \cap A_{i+1} = \varnothing$ for $i = 0, 1, 2, ..., n-1$.

(iii) $x_i \in A_{i-1}$ if x_i is a right accumulation point but not a left accumulation point; $x_i \notin A_{i-1} \cup A_i$ if x_i is an accumulation point from both sides; $x_i \in A_i$ if x_i is an isolated point and $\sigma_i = (-1)^i$; and $x_i \in A_{i-1}$ if x_i is an isolated point and $\sigma_i = (-1)^{i-1}$.

Then Σ is partially ordered by inclusion. For each $A \in \Sigma$, let $f_A \in C_0(Q)$ be a continuous function satisfying the properties that $f_A = f$ on A, $\|f_A\| = 1$, and f_A alternates at $(n-1)$ points $x_1^A, x_2^A, ..., x_{n-1}^A$, where $x_i^A = x_i$ if x_i is an isolated point. For each $A \in \Sigma$, let $g_A \in N$ be such that $\|f_A - g_A\| = d(f_A, N)$ and $f_A - g_A$ equioscillates at $(n+1)$ points. Then for each $A \in \Sigma$

$$\|g_A\| \leqslant \|f_A\| + d(f_A, N) \leqslant 2.$$

It will be shown that, for each $\varepsilon > 0$, there is $B_\varepsilon \in \Sigma$ such that for each $A \in \Sigma$, if $B_\varepsilon \subseteq A$, then $1 \geqslant d(f_A, N) > 1 - \varepsilon$. Either Q is not compact or $\{x_1, ..., x_{n-1}\}$ are not all isolated points. If Q is not compact then since $\dim N < \infty$ and $\|g_A\| \leqslant 2$ for each $A \in \Sigma$, it follows that there is a compact subset C of Q such that if $x \notin C$ and $A \in \Sigma$, then $|g_A(x)| < \varepsilon$. Let $B_\varepsilon \in \Sigma$ be such that $B_\varepsilon \setminus C \neq \varnothing$. Then B_ε satisfies the required properties. If $\{x_1, ..., x_{n-1}\}$ are not all isolated points, then there is $1 \leqslant i_0 \leqslant n-1$ such that x_{i_0} is an accumulation point. Since $\dim < \infty$ and $\|g_A\| \leqslant 2$ for each $A \in \Sigma$, it follows that there is a neighborhood U of x_{i_0} such that for each $x \neq y$ in U and $A \in \Sigma$, $|g_A(x) - g_A(y)| < \varepsilon$. Let $B_\varepsilon \in \Sigma$ be such that $B_\varepsilon = B_0 \cup \cdots \cup B_{i_0-1} \cup B_{i_0} \cup \cdots \cup B_{n-1}$, with $B_{i_0-1} \cap U \neq \varnothing$ and $B_{i_0} \cap U \neq \varnothing$. Then since g_A changes sign in U for each $B_\varepsilon \subseteq A \in \Sigma$, it follows that B_ε has the required properties.

By Lemma 2.6 if $B_\varepsilon \subseteq A \in \Sigma$, then there exists $z_A \notin \{x_1^A, ..., x_{n-1}^A\}$ such that $|g_A(z_A)| > 1 - \varepsilon$. Furthermore. it is obvious that g_A has the required sign on A. So since the set $\{x \in Q;\ |g_A(x)| \geqslant \frac{1}{2}$ for some $A \in \Sigma\}$ is compact, it follows that the net $\{(g_A, z_A);\ A \in \Sigma\}$ has a cluster point (g_0, z_0) in $N \times Q$, for which g_0 satisfies properties (1) and (2) and $|g_0(z_0)| \geqslant 1$. To show that g satisfies property (3) it is enough to show that $z_0 \notin \{x_1, ..., x_{n-1}\}$.

Assume that $z_0 \in \{x_1, ..., x_{n-1}\}$. Then if z_0 is not an isolated point in Q, it follows that $g_0(z_0) = 0$, which is a contradiction. If z_0 is an isolated point in Q, then since z_0 is a cluster point for the set $\{z_A; A \in \Sigma\}$, it follows that $z_0 = z_A$ for some $A \in \Sigma$. But then $z_0 \notin \{x_1, ..., x_{n-1}\}$.

2.8. Lemma. *Let Q be a locally compact totally ordered space, and let N be an n-dimensional subspace of $C_0(Q)$. If N has the property wc_4, then N is a weak Chebyshev subspace of $C_0(Q)$.*

Proof. By Proposition 1.2, it is enough to show that N has the property wc_3. Let $x_1 < x_2 < \cdots < x_{n-1}$ be $(n-1)$ points in Q. It will be shown by induction that for each $k \leqslant n-1$, and each $\{\sigma_{k+1}, ..., \sigma_{n-1}\}$ with $\sigma_i = \pm 1$, there is $g \in N$ satisfying the following properties:

(1) $g(x_i) = 0$ for $i = 1, 2, ..., k$.

(2) $\sigma_i g(x_i) \geqslant 0$ for $i = k+1, ..., n-1$.

(3) $(-1)^i g(x) \geqslant 0$ for $x \in (x_i, x_{i+1})$ and $i = 0, 1, ..., n-1$, where $x_0 = -\infty$ and $x_n = \infty$.

(4) There is $x \in Q \setminus \{x_1, ..., x_{n-1}\}$ such that $g(x) \neq 0$.

If this is true then N has the property wc_3.

For $k = 1$, by Lemma 2.7, there are g_1, g_2 in N satisfying properties (2), (3), and (4) with $g_1(x_1) \leqslant 0$ and $g_2(x_1) \geqslant 0$. If g_1 or g_2 is equal to zero at x_1 then the hypothesis is true for $k = 1$. Otherwise, let

$$g = \frac{g_1}{|g_1(x_1)|} + \frac{g_2}{|g_2(x_1)|}.$$

Then g satisfies properties (1), (2), and (3) and since $g_1(x)\ g_2(x) \geqslant 0$ for each $x \notin \{x_1, ..., x_{n-1}\}$, it follows that g satisfies property (4).

Assume that the hypothesis is true for some $k \leqslant n-2$. It will be shown that it is true for $k+1$.

Since the hypothesis is true for k, it follows that there are g_1 and g_2 in N satisfying properties (3) and (4) such that

$$\begin{aligned} &g_1(x_i) = g_2(x_i) && \text{for} \quad i = 1, 2, ..., k, \\ &g_1(x_{k+1}) \leqslant 0, && g_2(x_{k+1}) \geqslant 0, \\ &\sigma_i g_j(x_i) \geqslant 0 && \text{for} \quad i = k+2, ..., n-1 \text{ and } j = 1, 2. \end{aligned}$$

Using the same approach as that in the case when $k = 1$, one can show that the hypothesis is true for $k+1$.

2.9. THEOREM. *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional subspace of $C_0(Q)$. Then N has the property wc_4 if and only if N is a weak Chebyshev subspace.*

Proof. It follows from Lemmas 2.4 and 2.8.

2.10. THEOREM. *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional subspace of $C_0(Q)$. Then N is a weak Chebyshev subspace of $C_0(Q)$ if and only if it has one of the four equivalent properties wc_1, wc_2, wc_3, and wc_4.*

Proof. It follows from Proposition 1.2 and Theorem 2.9.

In the following example, it is shown that there is a compact totally ordered space Q that is not homeomorphic to any subset of the real numbers, such that for each $n \geqslant 1$, $C(Q)$ contains a weak Chebyshev subspace of dimension n. For the importance of this space one can see Brown [1, 2].

2.11. EXAMPLE. Let Q be the set $([0, 1]) \times \{0, 1\}) \backslash \{(0, 0), (1, 1)\}$, and let $\leqslant$ denote the lexicographic ordering on Q; that is,

$$(a, b) \leqslant (c, d) \qquad \text{if and only if either } a < c \text{ or } a = c \text{ and } b \leqslant d.$$

By Brown [2], the totally ordered space Q is compact and separable. Furthermore, it is not homeomorphic to any subset of the real numbers, and for each $(a, 0) \in Q$ the function f_a defined by

$$f_a(x) = \begin{cases} 1 & \text{if } x \leqslant (a, 0) \\ 0 & \text{if } x > (a, 0) \end{cases}$$

is continuous.

Let $n \geqslant 1$ be a positive integer, let $0 < a_1 < \cdots < a_n < 1$ be n real numbers, and let N be the n-dimensional subspace of $C(Q)$ generated by $\{f_{a_1}, f_{a_2}, ..., f_{a_n}\}$. Then it follows from Karlin [5, Example (ii), p. 16] that N has the property wc_2, and thus N is an n-dimensional weak Chebyshev subspace of $C(Q)$.

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REFERENCES

1. A. L. BROWN, An extension of Mairhuber's theorem on metric projection and discontinuity of multivariate best uniform approximation, *J. Approx. Theory* **36** (1982), 156–172.
2. A. L. BROWN, Chebyshev subspaces of finite codimension in spaces of continuous functions, *J. Austral. Math. Soc. Ser. A* **26** (1978), 99–109.
3. F. DEUTSCH, G. NÜRNBERGER, AND I. SINGER, Weak Chebyshev subspaces and alternations, *Pacific J. Math.* **88** (1980), 9–31.
4. R. JONES AND L. KARLOVITZ, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approx. Theory* **3** (1970), 138–145.
5. S. KARLIN, "Total Positivity I," Stanford Univ. Press, Stanford, CA, 1968.

6. G. NÜRNBERGER, Nonexistence of continuous selections of the metric projection and weak Chebyshev systems, *SIAM. J. Math. Anal.* **11**, No. 3 (1980), 460–467.
7. I. SCHOENBERG AND C. YANG, On the unicity of solutions of problems of best approximation, *Ann. Mat. Pura Appl.* **54** (1961), 1–12.
8. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag, Berlin, 1970.
9. R. ZIELKE, "Discontinuous Chebyshev Systems," Lecture Notes in Mathematics, Vol. 707, Springer-Verlag, Berlin/Heidelberg/New York, 1979.